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Subduction coefficients of Brauer algebras and Racah coefficients of O(n) and $Sp(2m)$: I. Subduction coefficients

Feng Pan† and Lianrong Dai‡

† CCAST (World Laboratory), PO Box 2375, Beijing 100080, People's Republic of China
and
Department of Physics, Liaoning Normal University, Dalian 116029, People's Republic of China
‡ Institute of High Energy Physics, Academia Sinica, PO Box 918-4-1, 100039 Beijing, People's Republic of China

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Abstract. Irreducible representations of Brauer algebras are discussed in the non-standard basis. A method for evaluating subduction coefficients (SDCs), i.e. the transformation coefficients between standard and non-standard bases of Brauer algebras, is outlined. Non-trivial SDCs of $D_f(n)$ for $f \leqslant 5$ are derived. Racah coefficients of O(n) and $Sp(2m)$ can be derived from subduction coefficients of Brauer algebras $D_f(n)$ by using the Schur–Weyl duality relation between $D_f(n)$ and O(n) or $Sp(2m)$.

1. Introduction

Brauer algebras [1, 2] $D_f(n)$, which are similar to the group algebra of the symmetric group S_f related to the decomposition of f-rank tensors of the general linear group $GL(n)$, are the centralizer algebras of the orthogonal group O(n) or the sympletic group $Sp(2m)$ when $n = -2m$. More precisely, if G is the orthogonal group O(n) or the sympletic group $Sp(2m)$, the corresponding centralizer algebra $B_f(G)$ are quotients of Brauer algebras $D_f(n)$ and $D_f(-2m)$, respectively [2, 3]. Hence, the duality relation between $D_f(n)$ and O(n) or $Sp(2m)$ is the same as the Schur–Weyl duality relation between S_f and $GL(n)$. Recently, irreducible representations of $D_f(n)$ in the standard basis, i.e. the basis adapted to the chain $D_f(n) \supset D_{f-1}(n) \supset \cdots \supset D_2(n)$, have been constructed by using the induced representation and the linear equation method [4], and more elaborately by Leduc and Ram using the so-called ribbon Hopf algebra approach [5].

On the other hand, Racah coefficients of classical Lie groups are of importance in many physical problems, which is apparent from the work of Kramer [6], Moshinsky and Chacón [7], Hecht [8], Le Blanc and Hecht [9], Alisăuskas [10], Judd *et al* [11], and so on.

In this series of papers, we shall use the Schur–Weyl duality relation between $D_f(n)$ and O(n) or $Sp(2m)$ to derive Racah coefficients of O(n) and $Sp(2m)$ from subduction coefficients of $D_f(n)$. This group-algebraic approach for evaluating Racah coefficients is similar to that used by Kramer [12] and Chen *et al* [13] to derive Racah coefficients of the unitary group $U(n)$ from subduction coefficients of the symmetric group S_f.

In section 2 we will briefly review the irreducible representations of Brauer algebra $D_f(n)$ in the standard basis. Then the non-standard basis adapted to the chain $D_f(n) \supset D_{f_1}(n) \times D_{f_2}(n)$ is expanded in terms of the standard ones. The expansion coefficients are called the subduction coefficients (SDCs), or the transformation coefficients between

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the standard and non-standard bases of $D_f(n)$. In section 3, we will present an effective procedure for evaluating these SDCs. SDC tables of $D_f(n)$ for $f \leqslant 5$ will be given in section 4, which are important in deriving Racah coefficients of $\mathrm{O}(n)$ and $Sp(2m)$.

2. Brauer algebra $D_f(n)$ in the non-standard basis

$D_f(n)$ is defined algebraically by $2f-2$ generators $\{g_1, g_2, \ldots, g_{f-1}, e_1, e_2, \ldots, e_{f-1}\}$ with the following relations:

$$\begin{aligned} &g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \\ &g_i g_j = g_j g_i \qquad |i-j| \geqslant 2 \\ &e_i g_i = e_i \\ &e_i g_{i-1} e_i = e_i. \end{aligned} \tag{1a}$$

Using the above-defined relations, we can obtain the following relations which are also useful for our purposes:

$$\begin{aligned} &e_i e_j = e_j e_i \qquad |i-j| \geqslant 2 \\ &e_i^2 = n e_i \\ &(g_i - 1)^2 (g_i + 1) = 0. \end{aligned} \tag{1b}$$

It can be easily seen that $\{g_1, g_2, \ldots, g_{f-1}\}$ generate a subalgebra S_f, i.e. $D_f(n) \supset S_f$. In the following, we always assume that n is an integer with $n \geqslant f-1$. In this case $D_f(n)$ is semisimple. Irreducible representations of $D_f(n)$ can be denoted by a Young diagram with $f, f-2, f-4, \ldots, 1$ or 0 boxes. An irrep of $D_f(n)$ with $f-2k$ boxes is denoted as $[\lambda]_{f-2k}$. The branching rule of $D_f(n) \downarrow D_{f-1}(n)$ is

$$[\lambda]_{f-2k} = \oplus_{[\mu]\leftrightarrow[\lambda]}[\mu] \tag{2}$$

where $[\mu]$ runs through all diagrams obtained by removing or (if $[\lambda]$ contains less than f boxes) adding a box to $[\lambda]$. Hence, the basis vectors of $D_f(n)$ in the standard basis can be denoted by

$$\left| \begin{matrix} [\lambda]_{f-2k} & C_f(n) \\ [\mu] & C_{f-1}(n) \\ \vdots & \vdots \\ [\rho] & C_{f-p+1}(n) \\ [\nu]_{f-p} & C_{f-p}(n) \end{matrix} \right) \equiv \left| \begin{matrix} [\lambda]_{f-2k} \\ [\mu] \\ \vdots \\ [\rho] \\ Y_M^{[\nu]} \end{matrix} \right) \tag{3}$$

where $[\nu]$ is identical to the same irrep of S_{f-p}, $Y_M^{[\nu]}$ is a standard Young tableau, and M can be understood either as the Yamanouchi symbols or the indices of the basis vectors in the so-called decreasing page order of the Yamanouchi symbols. Procedures for evaluating matrix elements of g_i, and e_i with $i = 1, 2, \ldots, f-1$ in the standard basis (3) have already been given in [4] and [5]. It is obvious that (3) is identical to the standard basis vectors of S_f when $k = 0$. In [4] explicit expressions for matrix representations of g_i and e_i in the standard basis were given for $f \leqslant 5$. Higher-dimensional results can also be derived by using the method given in [4] or by using Leduc and Ram's formula in [5].

An irrep of $D_f(n)$ is reducible with respect to its subalgebra $D_{f_1}(n) \times D_{f_2}(n)$ with $f_1 + f_2 = f$. The process of the reduction is denoted by

$$[\lambda]_{f-2k} \downarrow D_{f_1}(n) \times D_{f_2}(n) = \sum_{\lambda_1\lambda_2} \{\lambda_1\lambda_2\lambda\}([\lambda_1], [\lambda_2]). \tag{4}$$

We call this orthogonal subduced basis $D_f(n) \supset D_{f_1}(n) \times D_{f_2}(n)$ the non-standard basis of $D_f(n)$. The basis vectors of (4) are denoted by

$$\left|[\lambda]_{f-2k} \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1 & \rho_2 \end{matrix}\right\rangle \tag{5}$$

where $[\lambda_i]\rho_i$, $i = 1, 2$, can be understood as labels for the standard basis of $D_{f_1}(n)$, and $D_{f_2}(n)$, respectively, given by (3), $\tau = 1, 2, \ldots, \{\lambda_1\lambda_2\lambda\}$ is the multiplicity label needed in the reduction (4).

In order to determine matrix representations of $D_f(n)$ in the non-standard basis (5), we can expand the non-standard basis in terms of the standard ones given by (3):

$$\left|[\lambda]_{f-2k} \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1 & \rho_2 \end{matrix}\right\rangle = \sum_{\rho} \left|\begin{matrix}[\lambda]_{f-2k} \\ \rho\end{matrix}\right\rangle \left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho\end{matrix} \middle| \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1 & \rho_2 \end{matrix}\right\rangle. \tag{6}$$

The expansion coefficient is called the $[\lambda]_{f-2k} \downarrow [\lambda_1] \times [\lambda_2]$ SDC, or the transformation coefficient between the standard and non-standard bases of $D_f(n)$. The SDCs satisfy the following unitarity conditions:

$$\sum_{\lambda_2\rho_2\tau} \left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho\end{matrix} \middle| \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1 & \rho_2 \end{matrix}\right\rangle \left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho'\end{matrix} \middle| \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1 & \rho_2 \end{matrix}\right\rangle = \delta_{\rho\rho'} \tag{7a}$$

$$\sum_{\rho} \left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho\end{matrix} \middle| \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1 & \rho_2 \end{matrix}\right\rangle \left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho\end{matrix} \middle| \begin{matrix} \tau'[\lambda_1] & [\lambda_2'] \\ \rho_1 & \rho_2' \end{matrix}\right\rangle = \delta_{\lambda_2\lambda_2'}\delta_{\rho_2\rho_2'}\delta_{\tau\tau'}. \tag{7b}$$

Once the SDCs are determined, the matrix element $Q \in D_f(n)$ in the non-standard basis can easily be obtained with the results of those in the standard basis given in [4] and [5].

3. Evaluation of the SDCs

In [14, 15] we have already justified that the so-called linear equation method is effective in deriving SDCs as well as induction coefficients (IDCs) of Hecke algebras. This method can also be used to derive SDCs of $D_f(n)$.

Firstly, we assume that $\{g_1, g_2, \ldots, g_{f_1-1}, e_1, e_2, \ldots, e_{f_1-1}\}$, and $\{g_{f_1+1}, g_{f_1+2}, \ldots, g_{f-1}, e_{f_1+1}, e_{f_1+2}, \ldots, e_{f-1}\}$ are the generators of $D_{f_1}(n)$, and $D_{f_2}(n)$, respectively.

By applying $Q_i = g_i$ or e_i with $i = 1, 2, \ldots, f_1 - 1$ and $Q_j = g_j$ or e_j with $j = f_1 + 1, f_1 + 2, \ldots, f - 1$ to (6), and then multiplying the results from the left with

$$\left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho\end{matrix}\right|$$

we get two sets of linear equations

$$\sum_{\rho_1'} (Q_i)_{\rho_1'\rho_1} \left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho\end{matrix} \middle| \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1' & \rho_2 \end{matrix}\right\rangle = \sum_{\rho'} (Q_i)_{\rho\rho'} \left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho'\end{matrix} \middle| \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1 & \rho_2 \end{matrix}\right\rangle \tag{8a}$$

$$\sum_{\rho_2'} (Q_{j-f_1})_{\rho_2'\rho_2} \left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho\end{matrix} \middle| \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1 & \rho_2' \end{matrix}\right\rangle = \sum_{\rho'} (Q_j)_{\rho\rho'} \left\langle \begin{matrix}[\lambda]_{f-2k} \\ \rho'\end{matrix} \middle| \begin{matrix} \tau[\lambda_1] & [\lambda_2] \\ \rho_1 & \rho_2 \end{matrix}\right\rangle \tag{8b}$$

where $(Q_k)_{\rho\rho'}$ are matrix elements of Q_k in the corresponding standard basis. Linear relations or a part of the so-called intertwining relations among SDCs given in (8a) and (8b) together with the unitarity condition (7) are sufficient in solving these SDCs. Using these relations, we can obtain all SDCs for the given irreps $[\lambda]$, $[\lambda_1]$ and $[\lambda_2]$ when $[\lambda] \downarrow [\lambda_1] \times [\lambda_2]$ is multiplicity-free. In the multiplicity case, (8) gives linearly independent relations for the fixed multiplicity label. These relations are also sufficient in solving the SDCs with the fixed multiplicity label. However, the same relations hold for any other multiplicity labels. In order to resolve the multiplicity ambiguity, the SDCs with different multiplicity labels can be chosen to be orthogonal to each other. In this case the solution to the SDCs is not unique and depends on the phase convention. For example, SDCs of $D_6(n) \downarrow D_3(n) \times D_3(n)$ for the reduction $[321] \downarrow [21] \times [21]$ have already been given in [14] with $q = 1$ because SDCs for irreps of $D_f(n)$ with exactly f boxes are identical to those of symmetric groups S_f. In this paper, we will only derive SDCs of $D_f(n)$ for the irreps with $k \neq 0$ and $f \leqslant 5$ because $k = 0$ SDCs are the same as those of S_f, which are n-independent and have already been tabulated in [13]. In the following, we give an example to show how this method works.

Example. Derive SDC $\left\langle \begin{matrix}[2]\\ \rho\end{matrix} \right| \begin{matrix}[1] & [\lambda_2]\\ & \rho_2\end{matrix} \Big\rangle$ of $D_4(n) \downarrow D_1 \times D_3$, where $[\lambda_2] = [1]$, [21], or [3].

First, we rewrite (6) for these cases explicitly as

$$\begin{aligned}
&\left|[2]\,\begin{matrix}[1] & [1]\\ & [0]\end{matrix}\right\rangle = \sum_{i=1}^{6} a_i|i\rangle \qquad &&\left|[2]\,\begin{matrix}[1] & [1]\\ & [2]\end{matrix}\right\rangle = \sum_{i=1}^{6} b_i|i\rangle\\
&\left|[2]\,\begin{matrix}[1] & [1]\\ & [1^2]\end{matrix}\right\rangle = \sum_{i=1}^{6} c_i|i\rangle \qquad &&\left|[2]\,\begin{matrix}[1] & [1]\\ & [3]\end{matrix}\right\rangle = \sum_{i=1}^{6} d_i|i\rangle\\
&\left|[2]\,\begin{matrix}[1] & [1]\\ & [21]_1\end{matrix}\right\rangle = \sum_{i=1}^{6} f_i|i\rangle \qquad &&\left|[2]\,\begin{matrix}[1] & [1]\\ & [21]_2\end{matrix}\right\rangle = \sum_{i=1}^{6} h_i|i\rangle
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
&|1\rangle = \left|\begin{matrix}[2]\\ [3]\end{matrix}\right\rangle \qquad &&|2\rangle = \left|\begin{matrix}[2]\\ [21]_1\end{matrix}\right\rangle \qquad &&|3\rangle = \left|\begin{matrix}[2]\\ [21]_2\end{matrix}\right\rangle\\
&|4\rangle = \left|\begin{matrix}[2]\\ [1]\\ [2]\end{matrix}\right\rangle \qquad &&|5\rangle = \left|\begin{matrix}[2]\\ [1]\\ [1^2]\end{matrix}\right\rangle \qquad &&|6\rangle = \left|\begin{matrix}[2]\\ [1]\\ [0]\end{matrix}\right\rangle
\end{aligned} \tag{10}$$

and $a_i, b_i, \ldots, h_i$ are the corresponding SDCs.

Table 1. $D_3 \supset D_1 \times D_2$.

$D_3\backslash D_1 \times D_2$	[1] [0]	[1] [2]	[1] [1^2]
[1] [0]	$-\frac{1}{n}$	$-\sqrt{\frac{(n+2)(n-1)}{2n^2}}$	$\sqrt{\frac{n-1}{2n}}$
[1] [2]	$\sqrt{\frac{(n+2)(n-1)}{2n^2}}$	$\frac{n-2}{2n}$	$\sqrt{\frac{n+2}{4n}}$
[1] [1^2]	$\sqrt{\frac{n-1}{2n}}$	$-\sqrt{\frac{n+2}{4n}}$	$-\frac{1}{2}$

Table 2. $D_4 \supset D_2 \times D_2$.

$D_4\backslash D_2 \times D_2$	[2] [2]	[1^2] [1^2]	[0] [0]
[0] [1] [2]	1		
[0] [1] [1^2]		1	
[0] [1] [0]			1

Table 3. $D_4 \supset D_2 \times D_2$.

$D_4\backslash D_2 \times D_2$	$[1^2]$ $[2]$	$[1^2]$ $[1^2]$	$[0]$ $[2]$
$[2]$ $[21]_2$	$\sqrt{\frac{n-2}{2(n-1)}}$	$\sqrt{\frac{n}{2(n-1)}}$	
$[2]$ $[1]$ $[1^2]$	$-\sqrt{\frac{n}{2(n-1)}}$	$\sqrt{\frac{n-2}{2(n-1)}}$	
$[2]$ $[1]$ $[0]$			1
$D_4\backslash D_2 \times D_2$	$[2]$ $[2]$	$[2]$ $[1^2]$	$[2]$ $[0]$
$[2]$ $[3]$	$\sqrt{\frac{n-2}{3(n+2)}}$	$\sqrt{\frac{n+4}{3(n+2)}}$	$\sqrt{\frac{n+4}{3(n+2)}}$
$[2]$ $[21]_1$	$\sqrt{\frac{n+4}{6(n-1)}}$	$\sqrt{\frac{n-2}{6(n-1)}}$	$-\sqrt{\frac{2(n-2)}{3(n-1)}}$
$[2]$ $[1]$ $[2]$	$\sqrt{\frac{(n+4)(n-2)}{2(n+2)(n-1)}}$	$-\sqrt{\frac{n^2}{2(n+2)(n-1)}}$	$\sqrt{\frac{2}{(n+2)(n-1)}}$
$D_4\backslash D_2 \times D_2$	$[2]$ $[1^2]$	$[2]$ $[2]$	$[0]$ $[1^2]$
$[1^2]$ $[1]$ $[2]$	$\sqrt{\frac{n-2}{2(n-1)}}$	$-\sqrt{\frac{n}{2(n-1)}}$	
$[1^2]$ $[21]_1$	$\sqrt{\frac{n}{2(n-1)}}$	$\sqrt{\frac{n-2}{2(n-1)}}$	
$[1^2]$ $[1]$ $[0]$			1
$D_4\backslash D_2 \times D_2$	$[1^2]$ $[1^2]$	$[1^2]$ $[2]$	$[1^2]$ $[0]$
$[1^2]$ $[1^3]$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{n+2}{3n}}$	$\sqrt{\frac{n-2}{3n}}$
$[1^2]$ $[21]_2$	$\sqrt{\frac{n+2}{6(n-1)}}$	$\sqrt{\frac{(n-4)^2}{6n(n-1)}}$	$\sqrt{\frac{2(n^2-4)}{3n(n-1)}}$
$[1^2]$ $[1]$ $[1^2]$	$\sqrt{\frac{n-2}{2(n-1)}}$	$-\sqrt{\frac{(n^2-4)}{2n(n-1)}}$	$-\sqrt{\frac{2}{n(n-1)}}$

Applying g_2, e_2 to (9), and using (8a) and (8b) with the results of matrix representation in the standard basis given in [14], we obtain

$$a_5 = \sqrt{\frac{n}{n+2}}a_4 \qquad a_6 = \sqrt{\frac{2}{(n-1)(n+2)}}a_4 \tag{11a}$$

$$a_1 = a_2 = a_3 = 0$$

$$b_6 = -\sqrt{\frac{2(n+2)(n-1)}{(n-2)^2}}b_4 \qquad b_5 = -\sqrt{\frac{n(n+2)}{(n-2)^2}}b_4 \qquad b_3 = \sqrt{3}b_2 \tag{11b}$$

$$c_6 = -\sqrt{\frac{2(n-1)}{n+2}}c_4 \qquad c_5 = -\sqrt{\frac{n}{n+2}}c_4 \qquad c_1 = 0 \qquad c_2 = -\sqrt{3}c_3 \tag{11c}$$

$$d_6 = -\sqrt{\frac{2(n+2)(n-1)}{(n-2)^2}}d_4 \qquad d_5 = -\sqrt{\frac{n(n+2)}{(n-2)^2}}d_4 \qquad d_3 = \sqrt{3}d_2 \tag{11d}$$

$$f_6 = -\sqrt{\frac{2(n+2)(n-1)}{(n-2)^2}}f_4 \qquad f_5 = -\sqrt{\frac{n(n+2)}{(n-2)^2}}f_4 \qquad f_3 = \sqrt{3}f_2 \tag{11e}$$

Table 4. $D_5 \supset D_2 \times D_3$.

$D_5 \backslash D_2 \times D_3$	$[0]\ [1^3]$	$[1^2]\ [1^3]$	$[2]\ [1^3]$	$[1^2]\ [21]_1$	$[1^2]\ [21]_2$
$[1^3]\ [1^2]\ [1]\ [0]$	1				
$[1^3]\ [1^2]\ [1]\ [2]$			$\sqrt{\frac{n-3}{3(n-1)}}$		
$[1^3]\ [1^2]\ [1]\ [1^2]$		$\sqrt{\frac{n-3}{3(n-1)}}$	$\sqrt{\frac{(n-3)(n+2)}{2(n-1)^2}}$	$-\sqrt{\frac{(n-3)(n+2)}{6(n-1)^2}}$	
$[1^3]\ [1^2]\ [21]_1$			$\sqrt{\frac{n(n-3)}{3(n-1)(n-2)}}$		
$[1^3]\ [1^2]\ [21]_2$		$\sqrt{\frac{(n+2)(n-3)}{9(n-1)(n-2)}}$	$-\sqrt{\frac{(n-4)^2(n-3)}{6(n-2)(n-1)^2}}$	$-\sqrt{\frac{(n-3)(n+2)^2}{18(n-2)(n-1)^2}}$	
$[1^3]\ [1^2]\ [1^3]$		$-\sqrt{\frac{2(n-3)}{9(n-2)}}$		$\sqrt{\frac{(n-3)(n+2)}{3(n-2)(n-1)}}$	$\sqrt{\frac{(n-3)(n+2)}{9(n-2)(n-1)}}$
$[1^3]\ [1^4]$		$\sqrt{\frac{1}{6}}$			$\sqrt{\frac{n+2}{3(n-1)}}$
$[1^3]\ [211]_1$			$\sqrt{\frac{n}{3(n-2)}}$		
$[1^3]\ [211]_2$		$\sqrt{\frac{(n+2)}{9(n-2)}}$		$\sqrt{\frac{2}{3(n-2)(n-1)}}$	$\sqrt{\frac{2(n-4)^2}{9(n-2)(n-1)}}$
$[1^3]\ [211]_3$		$-\sqrt{\frac{(n+2)}{18(n-2)}}$		$\sqrt{\frac{4}{3(n-2)(n-1)}}$	$-\sqrt{\frac{(n-4)^2}{9(n-2)(n-1)}}$
$D_5 \backslash D_2 \times D_3$	**$[2]\ [21]_1$**	**$[2]\ [21]_2$**	**$[1^2]\ [1]_{1^2}$**	**$[1^2]\ [1]_2$**	**$[1^2]\ [1]_0$**
$[1^3]\ [1^2]\ [1]\ [0]$					
$[1^3]\ [1^2]\ [1]\ [2]$	$-\sqrt{\frac{n}{2(n-1)}}$	$\sqrt{\frac{n}{6(n-1)}}$			
$[1^3]\ [1^2]\ [1]\ [1^2]$			$\frac{1}{n-1}$	$-\sqrt{\frac{n+2}{n(n-1)^2}}$	$-\sqrt{\frac{2}{n(n-1)}}$
$[1^3]\ [1^2]\ [21]_1$	$\sqrt{\frac{n-2}{2(n-1)}}$	$\sqrt{\frac{n^2}{6(n-1)(n-2)}}$			
$[1^3]\ [1^2]\ [21]_2$			$\sqrt{\frac{n+2}{3(n-1)^2(n-2)}}$	$\sqrt{\frac{(n-4)^2}{3n(n-2)(n-1)^2}}$	$\sqrt{\frac{2(n^2-4)}{3n(n-1)}}$
$[1^3]\ [1^2]\ [1^3]$			$-\sqrt{\frac{2}{3(n-1)(n-2)}}$	$-\sqrt{\frac{2(n+2)}{3n(n-1)(n-2)}}$	$\sqrt{\frac{n-2}{3n}}$
$[1^3]\ [1^4]$			$\sqrt{\frac{n-3}{2(n-1)}}$		
$[1^3]\ [211]_1$		$-\sqrt{\frac{2(n-3)}{3(n-2)}}$			
$[1^3]\ [211]_2$			$-\sqrt{\frac{(n+2)(n-3)}{3(n-1)(n-2)}}$	$\sqrt{\frac{n-3}{3(n-2)(n-1)}}$	
$[1^3]\ [211]_3$			$-\sqrt{\frac{(n+2)(n-3)}{6(n-1)(n-2)}}$	$\sqrt{\frac{2(n-3)}{3(n-2)(n-1)}}$	

$$h_6 = \sqrt{\frac{2(n-1)}{(n+2)}}h_4 \qquad h_5 = -\sqrt{\frac{n}{n+2}}h_4 \qquad h_2 = -\sqrt{3}h_3. \tag{11f}$$

Table 5. $D_5 \supset D_2 \times D_3$.

$D_5 \backslash D_2 \times D_3$	[0] [3]	[2] [3]	[1^2] [3]	[1^2] $[21]_1$	[1^2] $[21]_2$
[3] [2] [1] [0]	1				
[3] [2] [1] [2]		$\sqrt{\frac{(n-2)(n+1)(n+6)}{3(n-1)(n+2)^2}}$			
[3] [2] [1] [1^2]			$-\sqrt{\frac{n+1}{3(n-1)}}$	$\sqrt{\frac{n-2}{6(n-1)}}$	$\sqrt{\frac{n-2}{2(n-1)}}$
[3] [2] $[21]_1$		$\sqrt{\frac{(n+1)(n+6)}{9(n+2)(n-1)}}$			
[3] [2] $[21]_2$			$\sqrt{\frac{(n-2)(n+1)}{3n(n-1)}}$	$-\sqrt{\frac{(n-2)^2}{6n(n-1)}}$	$\sqrt{\frac{n}{2(n-1)}}$
[3] [2] [3]		$\sqrt{\frac{2(n-2)(n+1)(n+6)}{9(n+2)^2(n+4)}}$			
[3] [4]		$\sqrt{\frac{n(n-2)}{6(n+2)(n+4)}}$			
[3] $[31]_1$		$\sqrt{\frac{n+6}{18(n+2)}}$			
[3] $[31]_2$		$\sqrt{\frac{n+6}{9(n+2)}}$			
[3] $[31]_3$			$\sqrt{\frac{n-2}{3n}}$	$\sqrt{\frac{2(n+1)}{3n}}$	
$D_5 \backslash D_2 \times D_3$	**[2] $[21]_1$**	**[2] $[21]_2$**	**[2] $[1]_{1^2}$**	**[2] $[1]_2$**	**[2] $[1]_0$**
[3] [2] [1] [0]					
[3] [2] [1] [2]	$-\sqrt{\frac{n(n-2)(n+1)}{6(n+2)(n-1)^2}}$	$-\sqrt{\frac{n(n-2)(n+1)}{2(n+2)(n-1)^2}}$	$-\sqrt{\frac{n}{(n+2)(n-1)^2}}$	$\frac{n-2}{(n+2)(n-1)}$	$-\sqrt{\frac{2}{(n+2)(n-1)}}$
[3] [2] [1] [1^2]					
[3] [2] $[21]_1$	$-\sqrt{\frac{n(n+1)}{18(n-1)^2}}$	$\frac{n-2}{n-1}\sqrt{\frac{n+1}{6n}}$	$\sqrt{\frac{n-2}{3n(n-1)^2}}$	$\sqrt{\frac{n-2}{3(n+2)(n-1)^2}}$	$\sqrt{\frac{2(n-2)}{3(n-1)}}$
[3] [2] $[21]_2$					
[3] [2] [3]	$-\sqrt{\frac{n(n-2)(n+1)}{9(n+2)(n-1)(n+4)}}$	$\sqrt{\frac{(n-2)(n+1)(n+4)}{3n(n+2)(n-1)}}$	$\sqrt{\frac{2(n+4)}{3n(n+2)(n-1)}}$	$\sqrt{\frac{2(n-2)^2}{3(n+2)^2(n-1)(n+4)}}$	$-\sqrt{\frac{n+4}{3(n+2)}}$
[3] [4]	$\sqrt{\frac{(n-2)(n+6)}{3(n-1)(n+4)}}$			$\sqrt{\frac{n(n+1)(n+6)}{2(n-1)(n+4)(n+2)}}$	
[3] $[31]_1$	$\sqrt{\frac{n}{9(n-1)}}$	$\sqrt{\frac{4}{3n(n-1)}}$	$-\sqrt{\frac{2(n-2)(n+1)}{3n(n-1)}}$	$-\sqrt{\frac{(n-2)(n+1)}{6(n-1)(n+2)}}$	
[3] $[31]_2$	$\sqrt{\frac{2n}{9(n-1)}}$	$-\sqrt{\frac{2}{3n(n-1)}}$	$\sqrt{\frac{(n-2)(n+1)}{3n(n-1)}}$	$-\sqrt{\frac{(n-2)(n+1)}{3(n-1)(n+2)}}$	
[3] $[31]_3$					

Applying g_3, e_3 to the d_i equation in (9), we get

$$d_2 = \sqrt{\frac{n+2}{3(n-2)}}d_4 \qquad d_1 = \sqrt{\frac{2(n-1)}{3(n+4)}}d_4. \tag{12a}$$

Table 6. $D_5 \supset D_1 \times D_4$.

$D_5 \backslash D_1 \times D_4$	$[1]\ [1^2]_{[1]0}$	$[1]\ [1^2]_{[1]1^2}$	$[1][1^2]_{[1]2}$	$[1]\ [1^2]_{[21]_1}$	$[1]\ [1^2]_{[21]_2}$
$[1^3]\ [1^2]\ [1]\ [2]$	$\sqrt{\frac{(n-1)(n+2)}{2n^2}}$	$\sqrt{\frac{n+2}{4n(n-1)^2}}$	$\frac{n-2}{2n(n-1)}$	$\sqrt{\frac{n-2}{4n(n-1)^2}}$	$\sqrt{\frac{(n+2)^2}{12n(n-2)(n-1)^2}}$
$[1^3]\ [1^2]\ [1]\ [1^2]$	$\sqrt{\frac{n-1}{2n}}$	$-\frac{1}{2(n-1)}$	$-\sqrt{\frac{n+2}{4n(n-1)^2}}$	$-\sqrt{\frac{n+2}{4(n-2)(n-1)^2}}$	$-\sqrt{\frac{n+2}{12(n-2)(n-1)^2}}$
$[1^3]\ [1^2]\ [1]\ [0]$	$-\frac{1}{n}$	$\sqrt{\frac{1}{2n(n-1)}}$	$-\sqrt{\frac{n+2}{2n^2(n-1)}}$	$-\sqrt{\frac{n+2}{2n(n-2)(n-1)}}$	$\sqrt{\frac{n+2}{6n(n-1)(n-2)}}$
$[1^3]\ [1^2]\ [21]_1$		$-\frac{\sqrt{n^2-4}}{2(n-1)}$	$\frac{\sqrt{n(n-2)}}{2(n-1)}$	$-\frac{1}{2(n-1)(n-2)}$	$-\frac{2n-5}{2\sqrt{3}(n-2)(n-1)}$
$[1^3]\ [1^2]\ [21]_2$		$-\frac{\sqrt{n^2-4}}{2\sqrt{3}(n-1)}$	$\frac{\sqrt{3n(n-2)}}{2(n-1)}$	$-\frac{\sqrt{3}}{2(n-1)(n-2)}$	$-\frac{2n-5}{6(n-2)(n-1)}$
$[1^3]\ [1^2]\ [1^3]$		$\sqrt{\frac{2(n-2)}{3(n-1)}}$			$-\sqrt{\frac{2(n+2)}{9(n-2)^2(n-1)}}$
$[1^3]\ [211]_1$				$\frac{\sqrt{(n-1)(n-3)}}{2(n-2)}$	$-\frac{\sqrt{(n-1)(n-3)}}{2\sqrt{3}(n-2)}$
$[1^3]\ [211]_2$				$\frac{\sqrt{3(n-1)(n-3)}}{2(n-2)}$	$\frac{\sqrt{(n-1)(n-3)}}{6(n-2)}$
$[1^3]\ [211]_3$					$\frac{\sqrt{8(n-1)(n-3)}}{3(n-2)}$
$[1^3]\ [1^4]$					

$D_5 \backslash D_1 \times D_4$	$[1]\ [1^2]_{[1^3]}$	$[1]\ [1^4]$	$[1][211]_1$	$[1]\ [211]_2$	$[1]\ [211]_3$
$[1^3]\ [1^2]\ [1]\ [2]$	$-\sqrt{\frac{n+2}{6n(n-1)(n-2)}}$	$\sqrt{\frac{(n-3)(n+2)}{8n(n-1)}}$	$-\sqrt{\frac{(n-3)(n-2)}{4n(n-1)}}$	$-\sqrt{\frac{(n-3)(n+2)^2}{12n(n-1)(n-2)}}$	$\sqrt{\frac{(n-3)(n+2)^2}{24n(n-2)(n-1)}}$
$[1^3]\ [1^2]\ [1]\ [1^2]$	$\sqrt{\frac{1}{6(n-1)(n-2)}}$	$-\sqrt{\frac{n-3}{8(n-1)}}$	$\sqrt{\frac{(n-3)(n+2)}{4(n-1)(n-2)}}$	$\sqrt{\frac{(n-3)(n+2)}{12(n-2)(n-1)}}$	$-\sqrt{\frac{(n-3)(n+2)}{24(n-2)(n-1)}}$
$[1^3]\ [1^2]\ [1]\ [0]$	$-\frac{1}{\sqrt{3n(n-2)}}$	$\sqrt{\frac{n-3}{4n}}$	$\sqrt{\frac{(n-3)(n+2)}{2n(n-2)}}$	$-\sqrt{\frac{(n-3)(n+2)}{6n(n-2)}}$	$\sqrt{\frac{(n-3)(n+2)}{12n(n-2)}}$
$[1^3]\ [1^2]\ [21]_1$	$-\sqrt{\frac{n+2}{6(n-1)(n-2)^2}}$	$\sqrt{\frac{(n-3)(n+2)}{8(n-1)(n-2)}}$	$\sqrt{\frac{n-3}{4(n-1)(n-2)^2}}$	$\frac{2n-5}{n-2}\sqrt{\frac{n-3}{12(n-1)}}$	$\frac{n+2}{n-2}\sqrt{\frac{n-3}{24(n-1)}}$
$[1^3]\ [1^2]\ [21]_2$	$\sqrt{\frac{n+2}{18(n-1)(n-2)^2}}$	$-\sqrt{\frac{(n-3)(n+2)}{24(n-1)(n-2)}}$	$\sqrt{\frac{3(n-3)}{4(n-1)(n-2)^2}}$	$-\frac{2n-5}{n-2}\sqrt{\frac{n-3}{36(n-1)}}$	$-\frac{n+2}{n-2}\sqrt{\frac{n-3}{72(n-1)}}$
$[1^3]\ [1^2]\ [1^3]$	$-\frac{1}{3(n-2)}$	$\sqrt{\frac{n-3}{12(n-2)}}$		$\frac{\sqrt{2(n-3)(n+2)}}{3(n-2)}$	$\frac{\sqrt{(n-3)(n+2)}}{6(n-2)}$
$[1^3]\ [211]_1$	$\sqrt{\frac{(n+2)(n-3)}{6(n-2)^2}}$	$\sqrt{\frac{n+2}{8(n-2)}}$	$\frac{1}{2(n-2)}$	$-\frac{1}{2\sqrt{3}(n-2)}$	$-\frac{3n-10}{\sqrt{24}(n-2)}$
$[1^3]\ [211]_2$	$-\sqrt{\frac{(n+2)(n-3)}{18(n-2)^2}}$	$-\sqrt{\frac{n+2}{24(n-2)}}$	$\frac{\sqrt{3}}{2(n-2)}$	$\frac{1}{6(n-2)}$	$\frac{3n-10}{\sqrt{72}(n-2)}$
$[1^3]\ [211]_3$	$\sqrt{\frac{(n+2)(n-3)}{36(n-2)^2}}$	$\sqrt{\frac{n+2}{48(n-2)}}$		$\frac{\sqrt{8}}{3(n-2)}$	$-\frac{3n-10}{12(n-2)}$
$[1^3]\ [1^4]$	$\sqrt{\frac{3(n-3)}{4(n-2)}}$	$-\frac{1}{4}$			$\sqrt{\frac{3(n+2)}{16(n-2)}}$

Using (11*d*), (12*a*) and the normalization condition in (7), we obtain

$$d_4 = \xi \sqrt{\frac{(n-2)^2(n+4)}{6(n-1)(n+2)^2}}. \tag{12b}$$

Table 7. $D_5 \supset D_1 \times D_4$.

$D_5 \backslash D_1 \times D_4$	$[1]\ [2]_{[1]0}$	$[1]\ [2]_{[1]1^2}$	$[1]\ [2]_{[1]2}$	$[1]\ [2]_{[3]}$	$[1]\ [2]_{[21]_1}$
[3] [2] [1] [2]	$\sqrt{\frac{(n-1)(n+2)}{2n^2}}$	$\sqrt{\frac{n+2}{4n(n-1)^2}}$	$\frac{(n-2)^2}{2n(n-1)(n+2)}$	$\sqrt{\frac{(n-2)^4}{6n^2(n-1)(n+4)(n+2)^2}}$	$\sqrt{\frac{(n-2)^3}{12n^2(n-1)^2(n+2)}}$
[3] [2] [1] [1^2]	$\sqrt{\frac{n-1}{2n}}$	$-\frac{1}{2(n-1)}$	$-\frac{n-2}{2(n-1)\sqrt{n(n+2)}}$	$-\sqrt{\frac{(n-2)^2}{6n(n+4)(n+2)(n-1)}}$	$-\sqrt{\frac{n-2}{12n(n-1)^2}}$
[3] [2] [1] [0]	$-\frac{1}{n}$	$\sqrt{\frac{1}{2n(n-1)}}$	$-\sqrt{\frac{(n-2)^2}{2n^2(n-1)(n+2)}}$	$-\sqrt{\frac{(n-2)^2}{3n^2(n+4)(n+2)}}$	$-\sqrt{\frac{n-2}{6n^2(n-1)}}$
[3] [2] $[21]_1$		$-\sqrt{\frac{3n(n-2)}{4(n-1)^2}}$	$-\sqrt{\frac{n^2(n-2)}{12(n-1)^2(n+2)}}$	$\sqrt{\frac{(n-2)^3}{18n^2(n+4)(n-1)(n+2)}}$	$-\frac{2n-1}{6n(n-1)}$
[3] [2] $[21]_2$		$\sqrt{\frac{n(n-2)}{4(n-1)^2}}$	$-\sqrt{\frac{n^2(n-2)}{4(n-1)^2(n+2)}}$	$\sqrt{\frac{(n-2)^3}{6n^2(n+4)(n-1)(n+2)}}$	$-\frac{2n-1}{2\sqrt{3}n(n-1)}$
[3] [2] [3]			$\sqrt{\frac{2n^2(n+4)}{3(n-1)(n+2)^2}}$	$\frac{(n-2)^2}{3n(n+2)(n+4)}$	$-\sqrt{\frac{2(n+4)(n-2)}{9n^2(n-1)(n+2)}}$
[3] [4]				$\sqrt{\frac{3(n+1)(n+2)(n+6)}{4n(n+4)^2}}$	
[3] $[31]_1$				$-\sqrt{\frac{(n^2-4)(n+1)}{36n^2(n+4)}}$	$\frac{\sqrt{8(n^2-1)}}{3n}$
[3] $[31]_2$				$-\sqrt{\frac{(n^2-4)(n+1)}{18n^2(n+4)}}$	$-\frac{\sqrt{n^2-1}}{6n}$
[3] $[31]_3$				$-\sqrt{\frac{(n^2-4)(n+1)}{6n^2(n+4)}}$	$-\frac{\sqrt{n^2-1}}{2\sqrt{3}n}$
$D_5 \backslash D_1 \times D_4$	$[1]\ [2]_{[21]_2}$	$[1]\ [31]_1$	$[1]\ [31]_2$	$[1]\ [31]_3$	$[1]\ [4]$
[3] [2] [1] [2]	$-\frac{\sqrt{n^2-4}}{2n(n-1)}$	$-\sqrt{\frac{(n+1)(n-2)^3}{24n^2(n-1)(n+2)}}$	$-\sqrt{\frac{(n+1)(n-2)^3}{12n^2(n-1)(n+2)}}$	$\sqrt{\frac{(n+1)(n^2-4)}{4n^2(n-1)}}$	$\sqrt{\frac{(n+1)(n+6)(n-2)^2}{8n(n-1)(n+2)(n+4)}}$
[3] [2] [1] [1^2]	$\sqrt{\frac{n-2}{4(n-1)^2n}}$	$\sqrt{\frac{(n+1)(n-2)}{24n(n-1)}}$	$\sqrt{\frac{(n+1)(n-2)}{12n(n-1)}}$	$-\sqrt{\frac{(n+1)(n-2)}{4n(n-1)}}$	$-\sqrt{\frac{(n+1)(n+6)}{8(n+4)(n-1)}}$
[3] [2] [1] [0]	$-\sqrt{\frac{n-2}{2n^2(n-1)}}$	$-\sqrt{\frac{(n+1)(n-2)}{12n^2}}$	$\sqrt{\frac{(n+1)(n-2)}{6n^2}}$	$\sqrt{\frac{(n+1)(n-2)}{2n^2}}$	$-\sqrt{\frac{(n+1)(n+6)}{4n(n+4)}}$
[3] [2] $[21]_1$	$-\frac{\sqrt{3}}{2n(n-1)}$	$-\sqrt{\frac{(n+1)(n-2)^2}{72n^2(n-1)}}$	$\sqrt{\frac{(n+1)(2n-1)^2}{36n^2(n-1)}}$	$\sqrt{\frac{3(n+1)}{4n^2(n-1)}}$	$\sqrt{\frac{(n+1)(n+6)(n-2)}{24n(n-1)(n+4)}}$
[3] [2] $[21]_2$	$\frac{1}{2n(n-1)}$	$-\sqrt{\frac{(n+1)(n-2)^2}{24n^2(n-1)}}$	$\sqrt{\frac{(n+1)(2n-1)^2}{12n^2(n-1)}}$	$-\sqrt{\frac{n+1}{4n^2(n-1)}}$	$-\sqrt{\frac{(n+1)(n+6)(n-2)}{8n(n-1)(n+4)}}$
[3] [2] [3]		$-\sqrt{\frac{(n+1)(n-2)^3}{36n^2(n+2)(n+4)}}$	$\sqrt{\frac{2(n-2)(n+1)(n+4)}{9n^2(n+2)}}$		$\sqrt{\frac{(n-2)^2(n+1)(n+6)}{12n(n+2)(n+4)^2}}$
[3] [4]		$\sqrt{\frac{3(n+6)(n-2)}{16n(n+4)}}$			$\frac{n-2}{4(n+4)}$
[3] $[31]_1$		$\frac{3n+2}{12n}$	$\frac{\sqrt{8}}{3n}$		$\sqrt{\frac{(n-2)(n+6)}{48n(n+4)}}$
[3] $[31]_2$	$-\frac{\sqrt{3(n^2-1)}}{2n}$	$\frac{3n+2}{6\sqrt{2}n}$	$-\frac{1}{6n}$	$\frac{\sqrt{3}}{2n}$	$\sqrt{\frac{(n-2)(n+6)}{24n(n+4)}}$
[3] $[31]_3$	$\frac{\sqrt{n^2-1}}{2n}$	$\frac{3n+2}{2\sqrt{6}n}$	$-\frac{1}{2\sqrt{3}n}$	$-\frac{1}{2n}$	$\sqrt{\frac{(n-2)(n+6)}{8n(n+4)}}$

Hence, all the d_i are known. Similarly, using $(11a)$ and the normalization condition for the

Table 8. $D_5 \supset D_3 \times D_2$.

$D_5 \backslash D_3 \times D_2$	$[1]_0\ [1^2]$	$[1]_2\ [1^2]$	$[1]_{1^2}\ [1^2]$	$[21]_1\ [2]$	$[21]_2\ [2]$	$[21]_1\ [1^2]$	$[21]_2\ [1^2]$	$[1^3]\ [2]$	$[1^3]\ [1^2]$	$[1^3]\ [0]$
$[1^3]\ [1^2]\ [1]\ [0]$	1									
$[1^3]\ [1^2]\ [1]\ [2]$		1								
$[1^3]\ [1^2]\ [1]\ [1^2]$			1							
$[1^3]\ [1^2]\ [21]_1$				$-\sqrt{\frac{n-1}{2(n-2)}}$		$\sqrt{\frac{n-3}{2(n-2)}}$				
$[1^3]\ [1^2]\ [21]_2$					$-\sqrt{\frac{n-1}{2(n-2)}}$		$\sqrt{\frac{n-3}{2(n-2)}}$			
$[1^3]\ [211]_1$				$\sqrt{\frac{n-3}{2(n-2)}}$		$\sqrt{\frac{n-1}{2(n-2)}}$				
$[1^3]\ [211]_2$					$\sqrt{\frac{n-3}{2(n-2)}}$		$\sqrt{\frac{n-1}{2(n-2)}}$			
$[1^3]\ [211]_3$								$-\sqrt{\frac{(n-6)^2}{8n(n-2)}}$	$-\sqrt{\frac{n+2}{8(n-2)}}$	$\sqrt{\frac{3(n-3)(n+2)}{4n(n-2)}}$
$[1^3]\ [1^2]\ [1^3]$								$\sqrt{\frac{(n-3)(n+2)}{2n(n-2)}}$	$-\sqrt{\frac{n-3}{2(n-2)}}$	$-\sqrt{\frac{3}{n(n-2)}}$
$[1^3]\ [1^4]$								$\sqrt{\frac{3(n+2)}{8n}}$	$\sqrt{\frac{3}{8}}$	$\sqrt{\frac{n-3}{4n}}$

a_i, we get

$$a_4 = \eta\sqrt{\frac{(n+2)(n-1)}{2n^2}} \tag{12c}$$

where ξ and η are overall phase factors which will be given in section 4. According to our phase convention, ξ and η are all taken to be $+1$. Finally, applying g_3 and e_3 to other equations in (9), we derive all the SDCs of this case. The results are listed in table 10.

4. SDCs of $D_f(n)$

In this section, we list some SDC tables derived by using the method outlined in the above section. Firstly, the following SDCs for $D_{f+1}(n) \supset D_f(n) \times D_1(n)$ are trivial:

$$\left\langle \begin{matrix} [\lambda] \\ \rho \end{matrix} \middle| [\lambda], \begin{matrix} [\lambda_1] \\ \rho_1 \end{matrix} \quad \begin{matrix} [1] \\ \end{matrix} \right\rangle = \delta_{\rho,[\lambda_1]\rho_1}. \tag{13}$$

Secondly, all SDC tables of symmetric groups S_f given in [13] are also the SDCs of $D_f(n)$ for the $D_f(n)$ irreps $[\lambda]_{f-2k}$ with $k = 0$. Hence, we will not re-tabulate them here. Other non-trivial SDCs of $D_f(n)$ derived by using our method are listed in tables 1–11.

Table 9. $D_5 \supset D_3 \times D_2$.

$D_5 \backslash D_3 \times D_2$	$[1]_0$ [2]	$[1]_2$ [2]	$[1]_{1^2}$ [2]	$[21]_1$ [2]	$[21]_2$ [2]
[3] [2] [1] [0]	1				
[3] [2] [1] [2]		1			
[3] [2] [1] [1^2]			1		
[3] [2] $[21]_1$				$\sqrt{\frac{n+1}{2n}}$	
[3] [2] $[21]_2$					$\sqrt{\frac{n+1}{2n}}$
[3] $[31]_2$				$\sqrt{\frac{n-1}{2n}}$	
[3] $[31]_3$					$\sqrt{\frac{n-1}{2n}}$
[3] $[31]_1$					
[3] [2] [3]					
[3] [4]					

$D_5 \backslash D_3 \times D_2$	$[21]_1$ [1^2]	$[21]_2$ [1^2]	[3] [0]	[3] [2]	[3] [1^2]
[3] [2] [1] [0]					
[3] [2] [1] [2]					
[3] [2] [1] [1^2]					
[3] [2] $[21]_1$	$-\sqrt{\frac{n-1}{2n}}$				
[3] [2] $[21]_2$		$-\sqrt{\frac{n-1}{2n}}$			
[3] $[31]_2$	$\sqrt{\frac{n+1}{2n}}$				
[3] $[31]_3$		$\sqrt{\frac{n+1}{2n}}$			
[3] $[31]_1$			$-\sqrt{\frac{3(n-2)(n+1)}{4n^2}}$	$\sqrt{\frac{(n+2)(n+6)}{8n^2}}$	$\sqrt{\frac{n-2}{8n}}$
[3] [2] [3]			$\sqrt{\frac{3(n+2)}{n^2(n+4)}}$	$\sqrt{\frac{(n-2)(n+1)(n+6)}{2n^2(n+4)}}$	$-\sqrt{\frac{(n+1)(n+2)}{2n(n+4)}}$
[3] [4]			$\sqrt{\frac{(n+1)(n+6)}{4n(n+4)}}$	$\sqrt{\frac{3(n^2-4)}{8n(n+4)}}$	$\sqrt{\frac{3(n+6)}{8(n+4)}}$

Table 10. $D_4 \supset D_1 \times D_3$.

$D_4 \backslash D_1 \times D_3$	[1] $[1]_0$	[1] $[1]_2$	[1] $[1]_{1^2}$	[1] [3]	[1] $[[21]_1$	[1] $[21]_2$
[2] $[1]_0$	$\frac{1}{n}$	$-\sqrt{\frac{(n-2)^2}{2n^2(n-1)(n+2)}}$	$\frac{1}{\sqrt{2n(n-1)}}$	$-\sqrt{\frac{n+4}{3(n+2)}}$	$\sqrt{\frac{n-2}{6(n-1)}}$	$\sqrt{\frac{n-2}{2(n-1)}}$
[2] $[1]_2$	$-\sqrt{\frac{(n-1)(n+2)}{2n^2}}$	$\frac{(n-2)^2}{2(n-1)n(n+2)}$	$\sqrt{\frac{n+2}{4n(n-1)^2}}$	$\sqrt{\frac{(n-2)^2(n+4)}{6(n+2)^2(n-1)}}$	$-\sqrt{\frac{(n-2)^3}{12(n+2)(n-1)^2}}$	$-\frac{\sqrt{n^2-4}}{2(n-1)}$
[2] $[1]_{1^2}$	$-\sqrt{\frac{n-1}{2n}}$	$-\sqrt{\frac{(n-2)^2}{4n(n+2)(n-1)^2}}$	$-\frac{1}{2(n-1)}$	$-\sqrt{\frac{n(n+4)}{6(n+2)(n-1)}}$	$\sqrt{\frac{n(n-2)}{12(n-1)^2}}$	$-\frac{\sqrt{n(n-2)}}{2(n-1)}$
[2] [3]		$\sqrt{\frac{2(n+4)n^2}{3(n-1)(n+2)^2}}$		$\frac{n-2}{3(n+2)}$	$\sqrt{\frac{2(n+4)(n-2)}{9(n-1)(n+2)}}$	
[2] $[21]_1$		$-\sqrt{\frac{(n-2)n^2}{12(n-1)^2(n+2)}}$	$\frac{-\sqrt{3n(n-2)}}{4(n-1)^2}$	$\sqrt{\frac{(n-2)(n+4)}{18(n-1)(n+2)}}$	$\frac{2n-1}{6(n-1)}$	$\frac{\sqrt{3}}{2(n-1)}$
[2] $[21]_2$		$-\sqrt{\frac{(n-2)n^2}{12(n-1)^2(n+2)}}$	$\sqrt{\frac{(n-2)n}{4(n-1)^2}}$	$\sqrt{\frac{(n-2)(n+4)}{6(n-1)(n+2)}}$	$\frac{2n-1}{(n-1)}$	$-\frac{1}{2(n-1)}$

The phase convention used for the SDCs of $D_f(n)$ is

$$\left\langle \begin{matrix} [\lambda] \\ \rho \end{matrix} \middle| [\lambda], \begin{matrix} [\lambda_1] \\ \rho_1 \end{matrix} \quad \begin{matrix} [\lambda_2] \\ \rho 2 \end{matrix} \right\rangle \Bigg|_{\rho=\min} > 0 \tag{14}$$

Table 11. $D_4 \supset D_1 \times D_3$.

$D_4 \backslash D_1 \times D_3$	[1] $[1]_0$	[1] $[1]_2$	[1] $[1]_{1^2}$	[1] $[1^3]$	[1] $[[21]_1$	[1] $[21]_2$
$[1^2]$ $[1]_0$	$-\frac{1}{n}$	$-\sqrt{\frac{n+2}{2n^2(n-1)}}$	$\frac{1}{\sqrt{2n(n-1)}}$	$\sqrt{\frac{n-2}{3n}}$	$\sqrt{\frac{n^2-4}{2n(n-1)}}$	$-\sqrt{\frac{n^2-4}{6n(n-1)}}$
$[1^2]$ $[1]_2$	$\sqrt{\frac{(n-1)(n+2)}{2n^2}}$	$\frac{n-2}{2n(n-1)}$	$\sqrt{\frac{n+2}{4n(n-1)^2}}$	$\sqrt{\frac{n^2-4}{6n(n-1)}}$	$-\sqrt{\frac{(n-2)^3}{4n(n-1)^2}}$	$-\frac{\sqrt{(n^2-4)(n+2)}}{12n(n-1)^2}$
$[1^2]$ $[1]_{1^2}$	$\sqrt{\frac{n-1}{2n}}$	$-\frac{\sqrt{n(n+2)}}{2n(n-1)}$	$-\frac{1}{2(n-1)}$	$-\sqrt{\frac{n-2}{6(n-1)}}$	$\frac{\sqrt{n^2-4}}{2(n-1)}$	$\sqrt{\frac{n^2-4}{12(n-1)^2}}$
$[1^2]$ $[1^3]$			$\sqrt{\frac{2(n-2)}{3(n-1)}}$	$\frac{1}{3}$		$\sqrt{\frac{2(n+2)}{9(n-1)}}$
$[1^2]$ $[21]_1$		$\frac{\sqrt{(n-2)n}}{2(n-1)}$	$-\sqrt{\frac{3(n^2-4}{12(n-1)^2}}$	$\sqrt{\frac{n+2}{6(n-1)}}$	$\frac{1}{2(n-1)}$	$\frac{2n-5}{2\sqrt{3}(n-1)}$
$[1^2]$ $[21]_2$		$\frac{\sqrt{3n(n-2)}}{2(n-1)}$	$\sqrt{\frac{(n^2-4}{12(n-1)^2}}$	$-\sqrt{\frac{n+2}{18(n-1)}}$	$\frac{\sqrt{3}}{2(n-1)}$	$-\frac{2n-5}{6(n-1)}$

where $\rho = \min$ means taking the index ρ as small as possible. The ordering of the index ρ is specified as follows. In the reduction $D_f(n) \downarrow D_{f-1}(n)$ with irrep $[\lambda] \downarrow [\mu]$, we always regard $\rho = [\bar{\mu}]\rho'$, where ρ' represents other indices in order to label irreps of $D_{f-1}(n)$, as smallest if $[\bar{\mu}]$ coincides with the same irrep of S_f. The same sub-ordering is then taken as that for symmetric groups given by [13] and [14]. For example, the basis vectors of $D_4(n)$ irrep [2] given in (10) are expressed in this ordering. Once the absolute phase is fixed, the relative phase among SDCs is determined uniquely by our linear equation method.

5. Conclusion

In this paper, the non-standard basis for Brauer algebras $D_f(n)$ is discussed, and the method for evaluating the SDCs of $D_f(n)$ is also presented. The SDCs of $D_f(n)$ for $f \leqslant 5$ are also tabulated. The SDCs of $D_f(n)$ are useful in evaluating Racah coefficients of O(n) and $Sp(2m)$ by using the Schur–Weyl duality relation between Brauer algebras and the corresponding orthogonal or sympletic groups, which will be discussed in our next paper.

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